Amortizing Bermudan Swaption Valuation

A valuation model is proposed for single-currency, Bermudan style fixed-for-floating interest rate swaptions with respect to the following features,

- exercise notification delay,
- amortizing fixed-leg payments, and
- time-dependent mean reversion.

We consider a swap specified as follows,

- single currency,
- notional, N ,
- reset point, T_i, for $i = 0, \ldots, M$, where $0 < T_0 < \ldots < T_M$,
- floating-leg payment, $N \, L(T_i; \Delta_i) \, \Delta_i$, at settlement time, T_{i+1} , for $i = 0, \ldots, M-1$, where
  - $\Delta_i = T_{i+1} - T_i$,
  - $L(T_i; \Delta_i) = \dfrac{1}{\Delta_i} \left(\dfrac{1}{P(T_i, T_{i+1})} - 1 \right)$ is the simple interest rate applicable at T_i for the accrual period, Δ_i ,
- fixed-leg payment, $N \, R_i \, \Delta_i$, at settlement time, T_{i+1} , for $i = 0, \ldots, M-1$, where R_i is a simple, annualized rate.

Note that the fixed-leg cashflows above can equivalently be viewed as cashflows from an amortizing notional but with constant fixed rate.

A Bermudan style swaption allows the holder to choose the entry point into the swap. Specifically, for $i = 1, \ldots, n$, let

- the reset point, T_{j_i}, be an exercise (or entry) point,
- τ_{j_i}, where $\tau_{j_i} \leq T_{j_i}$, be the notification time for exercise at T_{j_i}.

Here we assume that

- $0 < T_{j_1} < \ldots < T_{j_n}$ and
- $0 < \tau_{j_1} < \ldots < \tau_{j_n}$.

If notification is given at τ_{j_k}, where $k \in \{1,\ldots n\}$, then the Bermudan style swaption is exercised at T_{j_k}, which means that the following payments must be made at settlement time, T_{i+1}, for $i = j_k,\ldots, M-1$,

- floating-leg payment, $N\,L(T_i;\Delta_i)\,\Delta_i$,
- fixed-leg payment, $N\,R_i\,\Delta_i$.

Here we do *not* consider swaptions specified with any of the following features,

- up-front Libor payments (i.e., where Libor sets and is immediately paid),
- Libor accrual period different from reset accrual period.

We employ two methods, which are respectively for valuing single-currency, fixed-for-floating interest rate Bermudan style swaptions with features of the type. Both BK and AX methods are implementations of respective "disconnected" tree discretizations of a one factor Black-Karazinski (BK) risk-neutral short-rate process of the form below; moreover, the AX model incorporates certain enhancements to the BK model.

Let r denote the short-interest rate. We consider a short-interest rate process such that $\log r$ satisfies a risk-neutral SDE of the form,

$$d\log r_t = (\theta_t - a_t \log r_t)dt + \sigma_t dW_t, \qquad (3.1)$$

where

- a_t is a piecewise constant mean reversion rate,
- σ_t is a piecewise constant volatility function,
- θ_t is chosen to fit the initial term structure of discount factors,
- W_t is a standard Brownian motion.

A disconnected tree discretization of the short-rate process above is non-recombinant by design, but employs an interpolation scheme to approximate short-rate values at tree nodes along a time slice.

BAS did not have available any WM documentation describing the treatment of exercise notification delay in BK or AX models.

Calibrationis accomplished by matching, in a least squares sense, the model price against the market price for each respective European style payer swaption in a cache of calibration securities. Volatility break points are related to the forward start times of the respective swaptions in the calibration portfolio. Given a Bermudan swaption, we automatically select a corresponding calibration portfolio, which can be modified by the user if required.

Consider the Bermudan swaption defined in Section 2.1. Let r_t denote the short-interest rate at time t. We assume that the process, $\{\log r_t \mid 0 < t \le T_M\}$, satisfies a risk-neutral SDE of the Black-Karazinski form, (3.1).

We construct a trinomial tree to approximate the short-rate process, $\{r_t \mid 0 < t \le T_M\}$, based on the algorithm. Let

$$\Omega = \{t_i\}_{i=0}^{m},$$

where $0 = t_0 < ... < t_m = T_M$, be a partition of the interval, $[0, T_M]$, from valuation to the last settlement time. We assume that the respective volatility and mean reversion functions, σ_t and

a_t, are constant over each interval, $[t_{i-1}, t_i)$, for $i = 1, \ldots, m$. Moreover, we ensure the set of time slices, Ω, includes the following events,

- the reset point, T_i, for $i = 1, \ldots, M$,
- the notification point, τ_{j_i}, for $i = 1, \ldots, n$,
- the respective sets of volatility time and mean reversion time break points.
- Let T_{j_i}, where $i \in \{1, \ldots, n\}$, be an exercise point, and τ_{j_i} be the corresponding notification time. Consider an option to enter, at T_{j_i}, into a swap with right to pay fixed and receive float. Based on Section 2 specifications, this swap has value at notification time, τ_{j_i},
- $$\begin{aligned}\Phi_{\tau_{j_i}} &= \max\left[E^Q\left(\sum_{k=j_i}^{M-1} \frac{\left[L(T_k;\Delta_k) - R_k\right]\Delta_k}{e^{\int_{\tau_{j_i}}^{T_{k+1}} r(s)ds}} \middle| F_{\tau_{j_i}} \right), 0 \right], \\ &= \max\left(P(\tau_{i_j}, T_{i_j}) - \left(\sum_{k=j_i}^{M-1} P(\tau_{j_i}, T_{k+1}) R_k \Delta_k + P(\tau_{j_i}, T_M) \right), 0 \right),\end{aligned}$$
- where E^Q denotes the risk-neutral probability measure. The Bermudan swaption value is then given by
- $$\Omega = \max_{\tau \in \{\tau_{i_j}\}_{i=1}^{n}} E^Q\left(\frac{\Phi_\tau}{e^{\int_0^\tau r(s)ds}} \right)$$
- where τ denotes a stopping time.

Consider the Bermudan swaption specified in Section 2. Let

$$\Pi = \{\tau_1, \ldots, \tau_l\}$$

be a subset of the set of exercise points, $\Lambda = \{T_{i_j}\}_{i=1}^{n}$. We consider a portfolio of l European style payer swaptions. In particular, the i^{th} ($i = 1, \ldots, l$) swaption is specified as follows

- forward start time, τ_i,
- resets, $T_{j_i}, \ldots, T_{M-1}$, where $\tau_i = T_{j_i}$,

- payoff at τ_i, $\max\left(1-\left(X\sum_{k=j_i}^{M-1}\Delta_k P\left(T_{j_i},T_{k+1}\right)+P\left(T_{j_i},T_M\right)\right),0\right)$, where X is a strike level.

For each option in the portfolio above, we obtain from WM a corresponding implied Black's volatility. We then price the payer swaption above based on Black's analytical formula. Let P_i denote the price for the i^{th} ($i=1,\ldots,l$) swaption calculated using Black's model as described above.

We assume that the volatility function, σ_t, is constant over the respective periods, $[0,\tau_1)$, $[\tau_1,\tau_2),\ldots,[\tau_{l-1},T_M)$. Let σ_i, for $i=1,\ldots,l$, be the constant volatility value, which corresponds to the respective intervals $[0,\tau_1)$, $[\tau_1,\tau_2),\ldots,[\tau_{l-1},T_M)$. Furthermore let $bench(\vec{\sigma};i)$, for $i=1,\ldots,l$, denote the benchmark option price for the i^{th} swaption where $\vec{\sigma}=\left[\sigma_1,\ldots,\sigma_l\right]$. We seek to solve

$$\vec{F}(\vec{\sigma})=\begin{bmatrix} bench(\vec{\sigma};1)-P_1 \\ \vdots \\ bench(\vec{\sigma};l)-P_l \end{bmatrix}=\vec{0},$$

in a least squares sense, for the unknown, $\vec{\sigma}$; that is, we seek to minimize

$$\vec{F}(\vec{\sigma})^T\vec{F}(\vec{\sigma})=0$$

with respect to $\vec{\sigma}$. A necessary condition for a minimum is that

$$\left[\frac{\partial}{\partial\vec{\sigma}}\vec{F}(\vec{\sigma})^T\right]\vec{F}(\vec{\sigma})=\vec{0}\in R^l. \qquad (4.2.1)$$

We solve (4.2.1) for the unknown, $\vec{\sigma}$, using Newton's method based on our short-rate trinomial tree.

References:

https://finpricing.com/lib/FxCompound.html